

Simple random walk on the uniform infinite planar quadrangulation: Subdiffusivity *via* pioneer points

Itai Benjamini* and Nicolas Curien †

Abstract

We study the pioneer points of the simple random walk on the uniform infinite planar quadrangulation (UIPQ) using an adaptation of the peeling procedure of [3] to the quadrangulation case. Our main result is that, up to polylogarithmic factors, n^3 pioneer points have been discovered before the walk exits the ball of radius n in the UIPQ. As a result we verify the KPZ relation [27] in the particular case of the pioneer exponent and prove that the walk is subdiffusive with exponent less than $1/3$. Along the way, new geometric controls on the UIPQ are established.

Introduction

The goal of this work is to study the simple random walk on large random planar maps and especially on the Uniform Infinite Planar Quadrangulation (UIPQ). We show that the walk is dramatically affected by the geometry of the underlying random lattice and exhibits a behavior very different from the classical deterministic Euclidean setting. For example, we show that the walk is *subdiffusive*. Let us start by recalling the definition of the UIPQ.

A planar map is a proper embedding of a finite connected graph in the two-dimensional sphere, considered up to orientation-preserving homeomorphisms of the sphere. A *quadrangulation* is a planar map whose faces all have degree 4 (with the convention that if an edge lies entirely into a face then this edge is counted twice in the degree of the face). In this work, the maps that we will consider will systematically be rooted, that is, given with a distinguished oriented edge \vec{e} called the root of the map. The origin vertex of the root edge is called the origin of the map and is denoted by ρ .

The mathematical theory of random planar maps has been considerably growing over the last years motivated by the physics theory of 2D quantum gravity [1]. In particular, Miermont and Le Gall recently proved that a large class of random planar maps properly rescaled converge towards a universal continuous random surface called the *Brownian Map* [31, 36]. In this work, we choose a different perspective

*Weizmann Institute of Science

†École Normale Supérieure

and study *local limits* of random maps as introduced in [11]. If m, m' are two rooted maps, the local distance between m and m' is

$$d_{\text{map}}(m, m') = \left(1 + \sup\{r \geq 1 : \text{Ball}(m, r) = \text{Ball}(m', r)\}\right)^{-1},$$

where $\text{Ball}(m, r)$ denotes the map formed by the faces of m that have at least one vertex at distance strictly less than r from the origin ρ of m . The set of all finite quadrangulations is not complete for the metric d_{map} and we have to add infinite quadrangulations to make it complete, see [18] for more details. Let Q_n be a random rooted quadrangulation uniformly distributed over the finite set of all rooted quadrangulations with n faces. Krikun [28] proved the following convergence in distribution in the sense of d_{map}

$$Q_n \xrightarrow[n \rightarrow \infty]{(d)} Q_\infty, \quad (1)$$

where Q_∞ is a random infinite rooted quadrangulation called the *Uniform Infinite Planar Quadrangulation* (UIPQ). See also the pioneer work of Angel & Schramm [5] who introduced a similar object (the UIPT) in the triangulation case. It is believed that the UIPT and the UIPQ share the same large-scale properties. However, we chose to focus on the UIPQ rather than on the UIPT because of the existence of “nice” bijections between quadrangulations and simpler objects such as labeled trees [38] (these bijections do exist in the triangulation case but are less easy to manipulate). In particular, after the initial approach of Krikun [28], Chassaing & Durhuus [16] gave a Schaeffer-like construction of the UIPQ based on a random infinite tree with positive labels (which was shown to be equivalent to that of Krikun in [35]). The positivity constraint on the labels was relieved in [18] yielding to a third construction of the UIPQ (see Section 2).

The geometry of the UIPQ is very intriguing and is not completely understood. For instance, the UIPQ has a striking growth rate of r^4 [16, 33] but yet possesses separating cycles of linear length at all scales [28]. These isoperimetric inequalities heuristically suggest that the UIPQ has many folds and bottlenecks at all scales in which the nearest-neighbor simple random walk (SRW) could be trapped for a while, slowing it down. We will study this slowing effect by looking at particular points of the SRW called *pioneer points*. Let us define properly this notion.

Conditionally on Q_∞ , let $(X_n)_{n \geq 0}$ be a nearest-neighbor simple random walk on Q_∞ starting from the origin ρ . For any $k \geq 0$ we denote by \mathcal{R}_k the set of all faces of Q_∞ that are adjacent to the range $\{X_0, X_1, \dots, X_k\}$ of the walk up to time k . A time $k \geq 1$ is a *pioneer time* (in which case we say that X_k is a *pioneer point*) if X_k lies on the boundary of the only infinite component of $Q_\infty \setminus \mathcal{R}_{k-1}$ (the UIPQ has almost surely one end [28]). Our main result is:

Theorem 1 (Main result). *Let Q_∞ be the uniform infinite planar quadrangulation. Conditionally on Q_∞ , let $(X_n)_{n \geq 0}$ be a nearest-neighbor simple random walk on Q_∞ starting from ρ . We denote by P_1, P_2, \dots the pioneer points of $(X_n)_{n \geq 0}$. Then there exists a constant $\kappa > 0$ such that a.s. we eventually have*

$$n^{1/3} \log^{-\kappa}(n) \leq \max_{0 \leq k \leq n} d_{\text{gr}}(\rho, P_k) \leq n^{1/3} \log^\kappa(n).$$

We did not try to compute the best value of κ given by our proof and we do not have a precise guess for the correct logarithmic fluctuations. As a corollary of the proof of Theorem 1 we have:

Corollary 2 (Subdiffusivity). *With the notation of Theorem 1, there exists a constant $\kappa' > 0$ such that a.s. we eventually have*

$$d_{\text{gr}}(\rho, X_n) \leq n^{1/3} \log^{\kappa'}(n).$$

The simple random walk on the UIPQ thus has a *subdiffusive* behavior since it displaces much slower than the $n^{1/2}$ classical behavior of the simple random walk on $\mathbb{Z}^d, d \geq 1$. This phenomenon has first been suggested by Pierre-Gilles De Gennes [20] for the simple random walk on a critical percolation cluster: “*la fourmi dans un labyrinthe*”. This was rigorously proved by Kesten [26] for simple random walk on the infinite incipient cluster of critical two-dimensional Euclidean Bernoulli percolation (the exact exponent is still unknown) and finite variance critical Galton-Watson trees conditioned to survive (exponent $1/3$). This phenomenon has then been established for others models, see e.g. [6, 7, 17, 29]. We do not expect the $1/3$ exponent of Corollary 2 to be sharp and conjecture that $1/4$ is the correct value:

Conjecture 1. *The subdiffusivity exponent of the SRW on the UIPQ is $1/4$:*

$$\max_{0 \leq k \leq n} d_{\text{gr}}(\rho, X_k) \approx n^{1/4}.$$

See Section 5 for comments.

Usually, the road map to prove a subdiffusivity result is to estimate the volume growth and resistances in the graph. In our setting, evaluating resistances in the UIPQ remains a challenging problem. In particular, it is still open to show that the resistance between ρ and ∞ is infinite, in other words:

Conjecture 2. [5] *The UIPQ is almost surely recurrent.*

Rather than estimating resistances, the key to prove Theorem 1 is to use one of the main features of random planar maps: the *spatial Markov property* (Theorem 3). This property can roughly be stated as follows: Imagine that we explore a simply connected region of the UIPQ, then conditionally on the length of the boundary of this region, the remaining part of the UIPQ is independent of the explored region. The spatial Markov property of random planar triangulations has been used by Angel [3] to study several properties of the UIPT via the so-called *peeling process*. This is a clever random algorithm that discovers step-by-step the UIPT by revealing one face at a time, like “peeling an apple” [40]. Using the explicit transition probabilities of the peeling process of the UIPT, Angel [3] has obtained sharp estimates (up to polylogarithmic fluctuations) of the perimeter and the size of the triangulation discovered after n steps of peeling.

In this work, we will develop the same approach in the quadrangulation case and provide the analogs of the peeling estimates of Angel in the case of the UIPQ (Theorem 5). However, our tactics here does not consist in mimicking the proofs of [3] but rather to use the universality of the peeling process in order to translate geometric controls on the UIPQ (Section 3) into estimates on the peeling process.

Let us give a rough sketch of the proof of our main result. The idea is to discover the UIPQ along a simple random walk path using the peeling device. During this exploration, roughly speaking, only the pioneer points of the walk trigger the discovery of a new quadrangle. It then turns out that the boundary of the quadrangulation discovered after peeling n quadrangles of Q_∞ (or equivalently, after discovering $\approx n$ pioneer points of the walk) is of order $n^{2/3}$ (see Theorem 5). Now, by the spatial Markov property of the UIPQ, conditionally on the boundary of length $\approx n^{2/3}$, the remaining part of Q_∞ is independent of the revealed part. It has recently been proved in [19] that the typical distance between boundary points in a UIPQ with a boundary of perimeter $p = n^{2/3}$ is of order $\sqrt{p} = n^{1/3}$ which is the first glimpse at the $n^{1/3}$ appearing in Theorem 1.

Random planar maps are key tools in understanding Euclidean statistical physics systems *via* the Quantum Gravity approach, see e.g. [1]. Especially, the KPZ formula (Knizhnik, Polyakov and Zamolodchikov [27]) predicts relations between critical exponents of statistical mechanics models on a Euclidean lattice and the analogs on a random lattice (a random map). Duplantier and Sheffield rigorously proved the KPZ relations in a random geometry constructed from the Gaussian free field [23]. One missing link is the connection between random planar maps and the Gaussian free field, see the conjectures in [23, 39] and [8]. We propose a verification of the KPZ formula concerning pioneer points exponent of the simple random walk (Section 5). We also use the KPZ prediction on the disconnection exponent of the simple random walk on a random lattice to support Conjecture 1.

The paper is organized as follows. The next section introduces the spatial Markov property of the UIPQ, the peeling process, and the main estimates about it (Theorem 5). Section 2 presents the construction of [18] that we use in Section 3 to give new geometric lemmæ on the UIPQ. In particular, we study the vertex degrees in the UIPQ (Proposition 9) and provide uniform control on the volume growth (Proposition 11) and on the length of the separating cycles at a given height (Proposition 13). We then proceed to the (very short) proofs of Theorems 5 and 1 in Section 4. Unsurprisingly, the final section contains comments (in particular about the KPZ relation) and open questions.

Acknowledgments: We are grateful to Christophe Garban for a very stimulating discussion and for useful remarks. Thanks also go to Jean-François Le Gall for precious comments on an early version of this work and to Igor Kortchemski for a careful reading.

1 The peeling process

1.1 Spatial Markov property

This section is adapted from [5, Sections 4 and 5]. Since the proof are *mutatis mutandis* the same as in the triangulation case we leave them to the reader. Let us first introduce a few notions.

A quadrangulation with $k \geq 1$ holes is a (rooted) map with k distinct distinguished faces called the holes of the map, and such that all the non-distinguished faces have degree four. In the following, we will always assume that the boundaries of the holes are cycles with no self-intersection. Notice that by bipartiteness the holes must be of even degree. A quadrangulation q with one hole is also called a quadrangulation with a (simple) boundary. In this case, the degree $|\partial q|$ of the unique hole is called the perimeter of the map and its size $|q|$ is its number of inner faces. A quadrangulation with a simple boundary of perimeter $2p$ will also be called a quadrangulation of the $2p$ -gon. By convention, all the quadrangulations with a boundary that we consider here are rooted on the boundary in such a way that the hole is lying on the right-hand side of the distinguished oriented edge \vec{e} . Note that a quadrangulation of the 2-gon can be considered as a rooted quadrangulation (without hole) by contracting the unique face of degree 2.

Enumeration results. We write $\mathcal{Q}_{n,2p}$ for the set of all rooted quadrangulations of the $2p$ -gon with n inner faces. From [14, (2.11)] we read that for $p \geq 1$ and $n \geq 1$ we have

$$\#\mathcal{Q}_{n,2p} = \frac{3^{n-p}(3p)!(2n+p-1)!}{p!(2p-1)!(n-p+1)!(n+2p)!}. \quad (2)$$

In the case $p = 1$ and $n = 0$, the only element of $\mathcal{Q}_{0,2}$ is the quadrangulation with a simple boundary composed of one oriented edge. It is not “really” a quadrangulation but will be interpreted as follows (see the remark after Proposition 1.6 in [3]): A quadrangulation of the $2p$ -gon will often be used to close a hole of degree $2p$ in another quadrangulation, thus in the case $p = 1$ and $n = 0$ we close a hole of length 2 by gluing its two edges together. See Fig. 1.

More precisely, if q is a quadrangulation with holes we choose, once for all, a deterministic way of distinguishing an oriented edge on the boundary of each hole of q (such that the hole lies on the left of it). If q has a hole of perimeter $2p$ and if we are given a quadrangulation h of the $2p$ -gon, we can glue h inside the hole of q by identifying their boundaries (such that the oriented edge of the hole coincides with that of h). See Fig. 1. Let q, Q two quadrangulations with holes. We say that q is a submap of Q if Q can be obtained by filling some of the holes of q with quadrangulations with simple boundaries and we write

$$q \subset Q.$$

We say that q is *rigid* if two different ways of filling it lead to two different planar maps (see [5, Definition 4.7]). An easy adaptation of [5, Lemma 4.8] yields that any (rooted) quadrangulation with a simple boundary is rigid.

From (2), the asymptotic of $\#\mathcal{Q}_{n,2p}$ takes the form $\#\mathcal{Q}_{n,2p} \sim C_{2p} 12^n n^{-5/2}$ when $n \rightarrow \infty$, where C_{2p} a positive constant depending only on p . The $n^{-5/2}$ polynomial correction is typical of the enumeration of planar maps and plays a crucial role in the large scale structure of the UIPQ. In particular, the series

$$\sum_{n=0}^{\infty} \#\mathcal{Q}_{n,2p} 12^{-n},$$

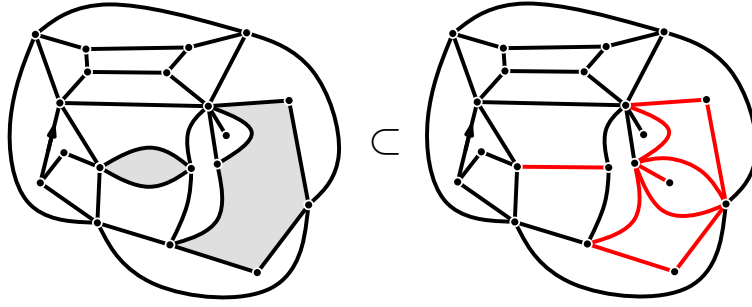


Figure 1: A (rigid) quadrangulation with holes and a way of filling it. Remark that the 2-gon on the left has been closed by a simple boundary of perimeter 2 and size 0.

is convergent and we denote its sum by Z_{2p} . Following [5, Definition 2.3] we define the *free distribution* on rooted quadrangulations of the $2p$ -gon as the probability measure ν_{2p} that assigns the weight $12^{-n} Z_{2p}^{-1}$ to each element of $\cup_{n \geq 0} \mathcal{Q}_{n,2p}$.

The convergence (1) can easily be extended to the case of quadrangulations with simple boundary: If $Q_{n,2p}$ is a uniform quadrangulation with size n and perimeter $2p$ then we have the convergence in distribution for d_{map}

$$Q_{n,2p} \xrightarrow[n \rightarrow \infty]{(d)} Q_{\infty,2p},$$

where $Q_{\infty,2p}$ is the UIPQ with simple boundary of perimeter $2p$ of UIPQ or the $2p$ -gon. This convergence is a simple consequence of (1), see [19]. We can now state the spatial Markov property of the UIPQ. Recall that almost surely the UIPQ (and more generally the UIPQ of the $2p$ -gon) has one end [28].

Theorem 3 (Spatial Markov Property). *Let q be a finite rigid (rooted) quadrangulation with $k+1$ holes of even degrees p_0, p_1, \dots, p_k such that the 0th hole is on the right of \vec{e} . Conditionally on the event $\{q \subset Q_{\infty,p_0}\}$ denote H_1, H_2, \dots, H_k the quadrangulations filling the first, second, third... holes of q in Q_{∞,p_0} . Then conditionally on $\{H_i \text{ is infinite}\}$, the quadrangulations H_1, \dots, H_k are independent and*

- (i) H_i has the same distribution as Q_{∞,p_i} ,
- (ii) for $j \neq i$, H_j is distributed according to ν_{p_j} .

1.2 The peeling algorithm

A growth algorithm for random maps, the *peeling process*, was first used heuristically by physicists (see [40] and [1, Section 4.7]). Angel [2, 3] then defined it rigorously and used it to study the volume growth and site percolation on the uniform infinite planar triangulation. We adapt his ideas to the context of the UIPQ.

The peeling process is a procedure that allows us to discover step-by-step the UIPQ by revealing one quadrangle at a time. Formally, we construct (on the same probability space) the uniform infinite planar quadrangulation Q_{∞} and a sequence of rooted quadrangulations with a boundary $Q_0 \subset Q_1 \subset \dots \subset Q_n \subset \dots \subset Q_{\infty}$,

such that for every $i \geq 0$, conditionally on Q_0, \dots, Q_i , the remaining part $Q_\infty \setminus Q_i$ has the same distribution as $Q_{\infty, |\partial Q_i|}$. The sequence is constructed inductively:

The quadrangulation Q_0 is the root edge of Q_∞ , which can be viewed as a quadrangulation with a boundary of perimeter 2. We write \mathcal{F}_n for the filtration generated by Q_0, \dots, Q_n . By the induction hypothesis, conditionally on \mathcal{F}_n , the remaining part $Q_\infty \setminus Q_n$ which is contained in the unique hole of Q_n has the same distribution as $Q_{\infty, |\partial Q_n|}$.

The conditional distribution of Q_{n+1} knowing \mathcal{F}_n can be described as follows. We first choose deterministically, or with the help of a randomized algorithm independent of $Q_\infty \setminus Q_n$, an edge e^* on ∂Q_n and re-root $Q_\infty \setminus Q_n$ at this edge. Since the choice of e^* is independent of $Q_\infty \setminus Q_n$ the newly rooted map still has the law of a UIPQ of the $|\partial Q_n|$ -gon. The peeling process then reveals the quadrangle in the remaining part $Q_\infty \setminus Q_n$ containing the edge e^* . Three cases may happen (see Fig. 2):

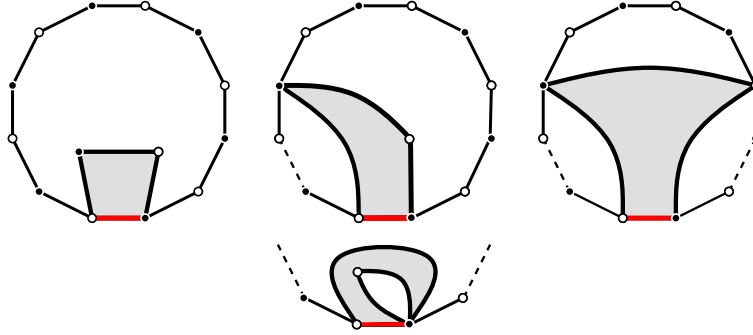


Figure 2: Illustration of the three cases that could happen when we reveal a new quadrangle in the unknown region. Notice that in the second case, two of the points on the boundary may be confounded.

- The revealed quadrangle has two vertices lying on the boundary ∂Q_n . In this case, we set Q_{n+1} to be the union of Q_n together with the revealed quadrangle. Hence Q_{n+1} is a quadrangulation with a boundary of length $|\partial Q_n| + 2$, and thanks to Theorem 3, conditionally on this event and on \mathcal{F}_n , the remaining quadrangulation $Q_\infty \setminus Q_{n+1}$ has the same distribution as $Q_{\infty, |\partial Q_{n+1}|}$.
- The quadrangle has three vertices lying on the boundary (two of these vertices might be identified). This quadrangle thus separates the remaining part into two quadrangulations $Q_{n,1}$ and $Q_{n,2}$ which are respectively quadrangulations of the p_1 -gon and p_2 -gon, such that $p_1 + p_2 = |\partial Q_n| + 2$. Since Q_∞ almost surely has one end, only one of this two components is infinite. For definiteness we argue on the event

$$A = \{Q_{n,1} \text{ is finite}, Q_{n,2} \text{ is infinite}\}.$$

Thanks to Theorem 3, conditionally on the revealed quadrangle, on A and on \mathcal{F}_n , $Q_{n,1}$ is distributed according ν_{p_1} and is independent of $Q_{n,2}$ which has the same distribution as Q_{∞, p_2} . We thus set Q_{n+1} to be the union of Q_n , of $Q_{n,1}$

and of the revealed quadrangle. Notice that Q_{n+1} is a quadrangulation with a boundary of perimeter p_2 and that conditionally on \mathcal{F}_{n+1} , $Q_\infty \setminus Q_{n+1}$ has the same distribution as $Q_{\infty, |\partial Q_{n+1}|}$.

- The quadrangle has its four vertices lying on the boundary of Q_n and separates the remaining part into three quadrangulations $Q_{n,1}$, $Q_{n,2}$ and $Q_{n,3}$. Similarly as in the preceding case, only one of these quadrangulations is infinite. We then set Q_{n+1} to be the union of these finite quadrangulations and of the revealed quadrangle and check that $Q_\infty \setminus Q_{n+1}$ has the desired law.

We stress the fact that there are many ways to do the peeling of Q_∞ according to the algorithm we use to choose the next quadrangle to reveal (provided that this choice is independent of the unknown part $Q_\infty \setminus Q_n$). Although the distribution of $Q_0, \dots, Q_n \dots$ may depend on the algorithm, the process $(|\partial Q_n|)_{n \geq 0}$ is actually a Markov chain whose distribution does not depend on the manner we revealed the squares in Q_∞ . Moreover the volume $|Q_n|$ of Q_n (its number of vertices) is obtained from $|Q_{n-1}|$ by filling with free quadrangulations of proper perimeters (and independent of the past) one or two holes whose perimeter only depend on the quadrangle revealed at time $n - 1$. Therefore a moment's thought shows that $(|\partial Q_n|, |Q_n|)_{n \geq 0}$ is a homogeneous Markov chain whose transition probabilities do not depend on the procedure chosen to do the peeling. Thus we have:

Lemma 4. *For any peeling procedure Q_0, \dots, Q_n, \dots the process $(|\partial Q_n|, |Q_n|)_{n \geq 0}$ has the same distribution.*

In [3], Angel explicitly computed the transition probabilities of the peeling in the case of the UIPT. Through a careful analysis of this chain, he proved that the boundary and the size of the triangulation obtained after n steps of peeling are respectively of order $n^{2/3}$ and $n^{4/3}$ up to polylogarithmic fluctuations. We will prove the same result in the case of the UIPQ. Before that, let us acquaint the reader with a useful notation.

In all this paper if $(Y_n)_{n \geq 0}$ is a random process indexed by \mathbb{N} with values in \mathbb{R}_+ , we write $Y_n \succeq n^\alpha$ resp. $Y_n \preceq n^\alpha$ for $\alpha > 0$ if there exists a constant $\kappa > 0$ such that we almost surely have

$$\lim_{n \rightarrow \infty} \frac{Y_n}{n^\alpha \log^{-\kappa}(n)} = \infty \quad \text{resp.} \quad \lim_{n \rightarrow \infty} \frac{Y_n}{n^\alpha \log^\kappa(n)} = 0.$$

If we have both $Y_n \preceq n^\alpha$ and $n^\alpha \preceq Y_n$ we write $Y_n \approx n^\alpha$. In words, $Y_n \approx n^\alpha$ means that almost surely Y_n grows like n^α up to polylogarithmic fluctuations. We also recall the classical Landau notation $x_n = O(y_n)$ (resp. $x_n = \Theta(y_n)$) if there exists a constant $0 < C < \infty$ (resp. $0 < c < C < \infty$) such that $x_n \leq Cy_n$ (resp. $cy_n \leq x_n \leq Cy_n$). We also denote $x_n \sim y_n$ if the quotient x_n/y_n goes to 1 as $n \rightarrow \infty$.

Theorem 5. *For any peeling procedure Q_0, \dots, Q_n, \dots we have*

$$|\partial Q_n| \approx n^{2/3} \quad \text{and} \quad |Q_n| \approx n^{4/3}. \quad (3)$$

Using the enumeration results of [14] it is possible to explicitly compute the transitions probabilities of the Markov chain $|\partial Q_n|$ (see [4]). It is also believable

that the arguments of [3] could be adapted to show Theorem 5. However, this is not the path we are about to follow. We propose a softer approach to this result. The idea is to get estimates on the peeling process *via* geometric estimates on the UIPQ. Indeed, because of Lemma 4 it is sufficient to prove Theorem 5 for *one* carefully chosen peeling algorithm. We will thus introduce an adaptation of the method proposed in [3] to analyze the volume growth in the UIPT: After establishing new results on the volume growth in Q_∞ , this process will be used in Section 4 to prove Theorem 5. These estimates will then be used with a second peeling coupled with a simple random walk on Q_∞ .

1.3 Peeling by layers

In this section we present a peeling process that discovers Q_∞ “layer after layer”. It is an adaptation of the peeling procedure of [3, Section 2]. Together with the geometric estimates of Section 3, this process will be used to deduce Theorem 5. In order to describe this peeling, we just have to tell how do we choose the next quadrangle to reveal in the process. We will then interpret it in a more geometric way.

Algorithm \mathcal{L} .

Algorithm \mathcal{L} “Layer”: Assume that $Q_n \subset Q_\infty$ is the quadrangulation with a boundary ∂Q_n containing the root edge of Q_∞ given by the peeling procedure \mathcal{L} at time n . The next quadrangle to reveal is chosen as follows. Pick an edge e^* on the boundary ∂Q_n such that one of its extremity minimizes $\{d_{\text{gr}}(\rho, x) : x \in \partial Q_n\}$ and reveal the quadrangle in $Q_\infty \setminus Q_n$ that contains e^* . Notice that there might be several edges satisfying this property, in this case, choose deterministically one of them.

This algorithm thus gives a way to peel the UIPQ. However, one must be careful with this procedure since one could have to use the “unknown” part $Q_\infty \setminus Q_n$ in order to compute the graph distance between the origin ρ and a point x on the boundary ∂Q_n . Recall that the choice of the edge to peel must not depend on $Q_\infty \setminus Q_n$ otherwise the law of the sequence $(|\partial Q_n|, |Q_n|)_{n \geq 0}$ might not be the same as a standard peeling process (Lemma 4). Fortunately, we will see (Proposition 6 (ii)) that if the preceding algorithm *has been used from the very first step* $n = 0$, then the graph distance between any point on ∂Q_n to ρ can be computed using Q_n only and thus the preceding algorithm yields to a true peeling process. Let us first introduce a piece of notation.

Interpretation. Recall that for $r \geq 1$, we denote by $\text{Ball}(Q_\infty, r)$ the quadrangulation contained in Q_∞ composed of the faces that have at least one vertex at distance strictly less than r from the origin $\rho \in Q_\infty$. In particular $\text{Ball}(Q_\infty, r) \subset \{u \in Q_\infty : d_{\text{gr}}(\rho, u) \leq r + 1\}$ in terms of vertex sets. By convention, the root edge of Q_∞ is considered as a face of degree two, so that $\vec{e} \subset \text{Ball}(Q_\infty, 1)$.

A moment’s thought shows that $\text{Ball}(Q_\infty, r)$ is a quadrangulation with holes, in particular the boundaries of the holes are cycles with no self-intersection. Since Q_∞ almost surely has one end, only one hole of $\text{Ball}(Q_\infty, r)$ corresponds to an infinite

quadrangulation of $Q_\infty \setminus \text{Ball}(Q_\infty, r)$. We denote the boundary of this hole by γ_r and called it the *separating cycle of ρ and ∞ in Q_∞ at height r* . Observe that this cycle is actually a simple path that alternatively visits vertices at distance r and $r + 1$ from the origin. We also denote by $\overline{\text{Ball}}(Q_\infty, r)$ the quadrangulation obtained from $\text{Ball}(Q_\infty, r)$ by filling all the finite holes of $\text{Ball}(Q_\infty, r)$ with their respective quadrangulations in Q_∞ . We call $\overline{\text{Ball}}(Q_\infty, r)$ the *hull* of the ball of radius r in Q_∞ . See Fig. 3 below.

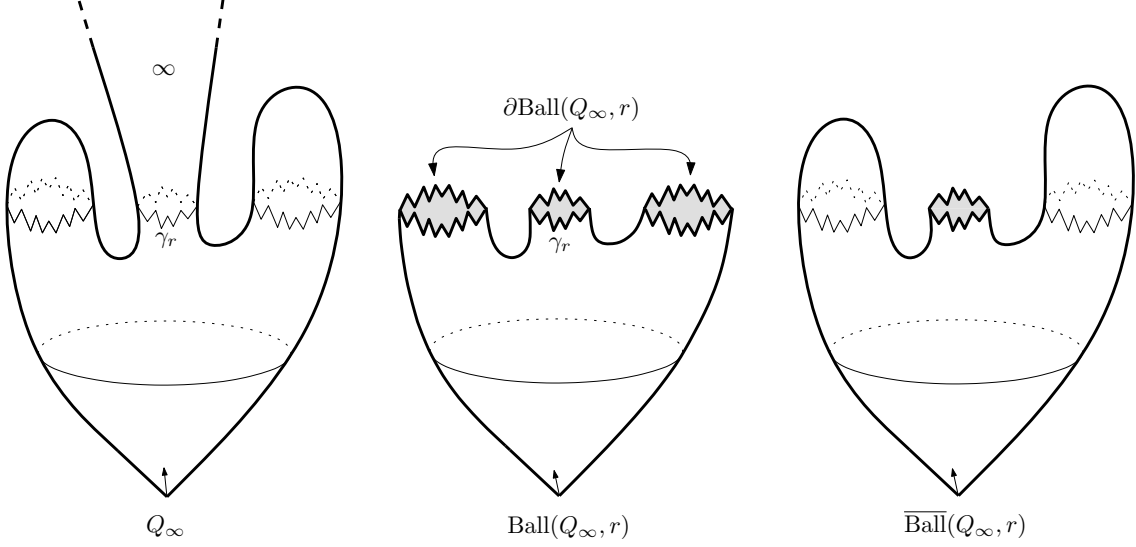


Figure 3: Illustration of $\text{Ball}(Q_\infty, r)$, γ_r , and $\overline{\text{Ball}}(Q_\infty, r)$ in Q_∞ .

The peeling process under Algorithm \mathcal{L} can geometrically be interpreted as follows: It roughly discovers Q_∞ layer after layer and stays very close to the cycles γ_k , $k \geq 0$ (see the figures of Section 4.7 in [1]). More precisely:

Proposition 6. *Let $Q_0 \subset Q_1 \subset Q_2 \subset \dots \subset Q_\infty$ be the successive quadrangulations with a boundary discovered using Algorithm \mathcal{L} . Then we have:*

- (i) *For every integer $r \geq 1$, let T_r be the first $n \geq 0$ such that $d_{\text{gr}}(\rho, \partial Q_n) \geq r$. Then $T_r < \infty$ a.s. and we have*

$$Q_{T_r} = \overline{\text{Ball}}(Q_\infty, r) \quad \text{and} \quad \partial Q_{T_r} = \gamma_r.$$

- (ii) *Futhermore, for any $n \geq 0$ and for any $x \in \partial Q_n$, the graph distance $d_{\text{gr}}(\rho, x)$ is measurable with respect to Q_n and thus the sequence $(|\partial Q_n|, |Q_n|)_{n \geq 0}$ has the distribution of a standard peeling process.*

Proof (Sketch). We prove the proposition by induction on $r \geq 1$. Let us first examine the case $r = 1$. We have $Q_0 = \vec{e}$ and start discovering some of the quadrangles that contain ρ . Notice that for any quadrangle adjacent to ρ , the graph distance from ρ of its vertices is either $\{0, 1, 0, 1\}$ or $\{0, 1, 2, 1\}$ by bipartiteness. Thus, when we discover a quadrangle containing ρ one can deduce the graph distance from ρ of its vertices by just looking at the quadrangulation discovered so far. Hence as long

as $n < T_1$ the graph distances of vertices of ∂Q_n to ρ are measurable with respect to Q_n .

Furthermore, all the quadrangles discovered for $n < T_1$ as well as the holes they created (which are filled-in during the process) are contained in $\overline{\text{Ball}}(Q_\infty, 1)$. We stop at T_1 when the origin ρ is not on the boundary of the current discovered quadrangulation Q_{T_1} . By the remarks above we have $Q_{T_1} \subset \overline{\text{Ball}}(Q_\infty, 1)$. The converse inclusion is deduced from the fact that the boundary of Q_{T_1} is composed of vertices that are alternatively at distance 1 and 2 from ρ . Hence $Q_{T_1} = \overline{\text{Ball}}(Q_\infty, 1)$ and $\gamma_1 = \partial Q_{T_1}$.

The general case $r \geq 2$ is pretty much the same and is safely left to the reader. \square

It follows from Proposition 6 that if r is the minimal distance in Q_∞ from a vertex in ∂Q_n to the origin ρ then we have

$$d_{\text{gr}}(\rho, x) \in \{r, r+1, r+2\}, \quad \text{and} \quad d_{\text{gr}}(\partial Q_n, \gamma_r) \leq 2. \quad (4)$$

1.4 Peeling along a simple random walk

We now describe another way of peeling Q_∞ . This one is coupled with a simple random walk and discovers the quadrangulation when necessary for the walk to make one more step. This peeling process is one of the keys in the proof of Theorem 1. We start with the formal definition of this algorithm and then interpret it in terms of pioneer points.

Algorithm \mathcal{W} .

Algorithm \mathcal{W} “Walk”: Let Q_∞ be the uniform infinite planar quadrangulation and conditionally on Q_∞ , let $(X_n)_{n \geq 0}$ be a nearest-neighbor simple random walk on Q_∞ starting from ρ . We do the peeling process each time we need it for the SRW to displace. More precisely, we define a sequence

$$\vec{e} = Q_0 \subset Q_1 \subset \dots \subset Q_n \subset \dots \subset Q_\infty$$

of quadrangulations with boundaries and two random non decreasing functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(0) = g(0) = 0$, $X_{g(k)} \in Q_{f(k)}$ for every $k \geq 0$, and whose evolution is described by induction as follows.

We have two cases. If the current position $X_{g(k)}$ of the simple random walk belongs to $\partial Q_{f(k)}$, then choose an edge e^* on $\partial Q_{f(k)}$ containing $X_{g(k)}$ and set $f(k+1) := f(k) + 1$ and $g(k+1) := g(k)$. The quadrangulation $Q_{f(k+1)}$ is the map obtained after the peeling associated with the edge e^* . If the current position $X_{g(k)}$ of the simple random walk belongs to $Q_{f(k)} \setminus \partial Q_{f(k)}$ then we set $f(k+1) := f(k)$ and $g(k+1) := g(k) + 1$.

Although this algorithm has an extra randomness due to the SRW, the edges chosen to be revealed in the peeling process are independent of the unknown part, and thus, thanks to Lemma 4 the process $(|\partial Q_n|, |Q_n|)_{n \geq 0}$ has the same law as the process obtained with Algorithm \mathcal{L} . We put

$$\tau_n := \sup\{g(k) : f(k) = n\}.$$

In words, τ_n is the number of steps made by the SRW inside Q_n . Note that $X_{\tau_n} \in \partial Q_n$. Since Q_{n+1} differs from Q_n by the peeling of an edge of ∂Q_n incident to X_{τ_n} we deduce by induction that for every $i \geq 0$ we have

$$d_{\text{gr}}(\partial Q_i, \{X_0, \dots, X_{\tau_i}\}) \leq 2. \quad (5)$$

Interpretation. Let us recall the definition of the pioneer points of $(X_n)_{n \geq 0}$. For any $k \geq 0$ we denote by \mathcal{R}_k the submap of Q_∞ formed by the faces that are adjacent to $\{X_0, X_1, \dots, X_k\}$. A moment's thought shows that \mathcal{R}_k is a quadrangulation with holes. Since Q_∞ almost surely has one end, only one of these holes corresponds to an infinite quadrangulation and we denote by $\overline{\mathcal{R}}_k$ the quadrangulation obtained from \mathcal{R}_k after filling all the finite holes with their respective quadrangulations in Q_∞ . Henceforth $\overline{\mathcal{R}}_k$ is a quadrangulation with a simple boundary called the *hull* of the range of X up to time k . Recall that for $k \geq 1$, the k^{th} step X_k of the SRW is a pioneer point (k is a pioneer time) if

$$X_k \in \partial \overline{\mathcal{R}}_{k-1}.$$

By convention $k = 0$ is a pioneer time.

Proposition 7. *The pioneer times of $(X_n)_{n \geq 0}$ are exactly the times $\{\tau_k : k \geq 0\}$.*

Proof (Sketch). We prove by induction the following property : (*) For all $k \geq 0$ such that we have $X_{g(k)} \in Q_{f(k)} \setminus \partial Q_{f(k)}$ then $Q_{f(k)}$ corresponds to the hull of the range of X up to time $g(k)$. Assume that this property holds for a certain k and let $n = f(k)$. Obviously, the property holds for all k such that $g(k) < \tau_n$. The time τ_n is thus a pioneer point and the peeling process is then triggered and we discover all the faces in $Q_\infty \setminus Q_n$ adjacent to X_{τ_n} (and fill the holes they create) until all of $\overline{\mathcal{R}}_{\tau_n}$ is revealed. At this point the SRW lies inside the current quadrangulation (not on its boundary) and the property (*) holds anew. Details are left to the reader. \square

Notice that the number of peeling steps is larger than or equal to the number of pioneer points visited so far minus one (recall that $t = 0$ is a pioneer time) because the discovery of a pioneer point automatically triggers a new step of peeling. However, the maximal number of steps that a pioneer point can trigger is obviously bounded above by its degree in Q_∞ , where the degree $\deg(u)$ of a vertex u is the number of edges adjacent to it. Hence we have

$$n + 1 \geq \#\{\text{pioneer times} \leq \tau_n\} \geq \frac{n}{\max\{\deg(u) : d_{\text{gr}}(\rho, u) \leq n\}}. \quad (6)$$

2 Construction of Q_∞ from a labeled tree

In Section 3 we gather some geometric estimates on the UIPQ. Most of the results depend on a Schaeffer-like construction of the UIPQ introduced in [18]. For sake of completeness, we briefly recall it here, the interested reader should consult [18] for more details.

2.1 The uniform infinite labeled tree (T_∞, ℓ)

We use the standard formalism for plane trees as found in [37]. A plane tree t is a tree given with an ancestor and an order for the children of any vertex $u \in t$. We use the same notation as [18]. In particular, the ancestor (or root) of a plane tree t is denoted by \emptyset and its size $|t|$ is its number of vertices. In the following, all the trees that we consider are plane trees. We denote the set of all rooted plane infinite trees with only one infinite geodesic (also called spine) by \mathcal{S} . A tree $t \in \mathcal{S}$ is thus composed of a unique infinite geodesic

$$\{\emptyset = s_0, s_1, s_2, \dots\},$$

and finite trees grafted to the left and to the right of each vertex s_i . The degree of a vertex $u \in t$, denoted by $\deg(u, t)$, is the number of edges adjacent to u in t . Such a tree can properly be drawn in plane without accumulation point of the vertices (and matching the ordering of the tree with the clockwise orientation of the plane). A *corner* of a vertex $u \in t$ is an angular sector between two consecutive edges in clockwise order around u (in a plane representation of t). A vertex of degree k thus has k corners. The contour of the tree $t \in \mathcal{S}$ is the bi-infinite sequence of corners

$$\{\dots, c_{-2}, c_{-1}, c_0, c_1, c_2, \dots\}$$

sorted in clockwise order where c_0 is the corner of the ancestor \emptyset where the tree t is rooted. If c_i and c_j are two distinct corners of t , we denote by $[c_i, c_j]$ the set of corners that are inbetween c_i and c_j for the contour order (notice that if $i \geq 0$ and $j \leq 0$ then $[c_i, c_j]$ is infinite). If c is a corner of t then $\mathcal{V}(c)$ is the vertex associated with c .

A *labeling* of a plane tree t is a collection $\{\ell(u) : u \in t\}$ of variables with values in \mathbb{Z} attached to each vertex of t such that $\ell(\emptyset) = 0$ and $\ell(u) - \ell(v) \in \{-1, 0, +1\}$ for any neighboring vertices $u, v \in t$. The label $\ell(c)$ of a corner c is the label of its vertex. We now define the notion of successor. Let (t, ℓ) be an infinite labeled tree in \mathcal{S} and let c_i be a corner of t . The successor of c_i is the first corner $\mathcal{S}(c_i)$ belonging to

$$\{c_{i+1}, c_{i+2}, \dots\} \cup \{\dots, c_{i-2}, c_{i-1}\}$$

such that $\ell(\mathcal{S}(c_i)) = \ell(c_i) - 1$. Note that if the vertex associated with c has a minimal label among t then c has no successor (this case will not show up in our setup).

We now present the random infinite labeled tree (T_∞, ℓ) which the UIPQ is constructed from. Firstly, T_∞ is a geometric critical Galton-Watson tree conditioned to survive (see [26, 34]). The distribution of T_∞ can roughly be described as follows: T_∞ has a unique spine (thus $T_\infty \in \mathcal{S}$) and the subtrees grafted to the left and to the right of each vertex of the spine are independent critical geometric Galton-Watson trees. See [18] for more details. We recall that if T is a critical geometric Galton-Watson (of parameter $1/2$) then we have

$$P(|T| = n + 1) = \frac{\text{Cat}(n)}{2 \cdot 4^n} \sim \frac{n^{-3/2}}{2\sqrt{\pi}}, \quad (7)$$

where $|T|$ is the number of vertices of T and $\text{Cat}(n) = \binom{2n}{n}/(n+1)$ is the n th Catalan number. Thus the random variable $|T|$ is in the domain of attraction of a completely asymmetric stable variable with parameter $1/2$.

We then label the tree T_∞ according to the following device: Conditionally on T_∞ , let $\{d_e : e \in \text{Edges}(T_\infty)\}$ be independent random variables uniformly distributed over $\{-1, 0, +1\}$ carried by the edges of T_∞ . This defines a labeled tree (T_∞, ℓ) where the label of a vertex is the sum of the d_e 's along its ancestral path towards the ancestor \emptyset . If (t, ℓ) is a labeled tree we set

$$\Delta(t, \ell) = \max\{|\ell(u)| : u \in t\}.$$

In the following, (T_∞, ℓ) always denotes the tree constructed above that we call the uniform infinite labeled tree. For $n \geq 0$, we denote by \mathcal{T}_n the (labeled) subtree obtained from (T_∞, ℓ) after pruning at the n^{th} vertex of the spine s_n , that is, we remove all the offspring of s_n (but we keep s_n). Recall that $|\mathcal{T}_n|$ is the number of vertices of \mathcal{T}_n . We also denote $\varnothing(\mathcal{T}_n) := \max\{\text{dist}(u, v) : u, v \in \mathcal{T}_n\}$ where $\text{dist}(\cdot, \cdot)$ is the graph distance in \mathcal{T}_n , its diameter. Recall the notation \preceq, \succeq and \approx from Section 1.2.

Proposition 8. *We have*

$$\begin{aligned} \varnothing(\mathcal{T}_n) &\approx n, \\ |\mathcal{T}_n| &\approx n^2, \end{aligned} \tag{8}$$

$$\Delta(\mathcal{T}_n) \approx n^{1/2}. \tag{9}$$

Proof (Sketch). These are pretty standard facts but we include a proof for sake of completeness. Let $n \geq 0$. The tree \mathcal{T}_n is composed of the first $n + 1$ vertices on the spine together with $2n$ independent critical geometric Galton-Watson trees grafted to the right-hand side and to the left-hand side of s_0, s_1, \dots, s_{n-1} (when there is no tree on one side of a vertex of spine we consider that we grafted the tree with a single vertex). Thus we have

$$|\mathcal{T}_n| = 1 - n + \sum_{i=1}^{2n} X_i \quad \text{and} \quad n \leq \varnothing(\mathcal{T}_n) \leq 2\left(n + \max_{1 \leq i \leq 2n} H_i\right)$$

where $X_1, H_1, X_2, H_2, \dots$ are respectively the size and the height of the $2n$ critical geometric Galton-Watson trees grafted on the n first vertices of the spine. Recall from (7) that we have $P(X_1 \geq n) = \Theta(n^{-1/2})$. Recall also the classical Kolmogorov's estimate $P(H_1 \geq n) \sim n^{-1}$. From the latter we easily deduce using Borel-Cantelli lemma that eventually $H_i \leq i \log^2(i)$ and thus $\varnothing(\mathcal{T}_n) \approx n$. Concerning the size $|\mathcal{T}_n|$, the analogue of the law of the iterated logarithm in the case of infinite variance (see [13, Section 3.9]) directly show that $S_n = \sum_{i=1}^{2n} X_i \approx n^2$ which implies $|\mathcal{T}_n| \approx n^2$.

Let us now turn to (9). Recall that conditionally on the tree structure of \mathcal{T}_n , the labels evolve along the branches of \mathcal{T}_n as a random walk $(Z_k)_{k \geq 0}$ whose increments are uniform in $\{-1, 0, +1\}$. Looking at the labels of s_0, \dots, s_{n-1} we deduce that $\Delta(\mathcal{T}_n) \geq \max_{0 \leq i \leq n-1} |\ell(s_i)|$ which gives the lower bound $\Delta(\mathcal{T}_n) \succeq n^{1/2}$. For the upper bound, we have

$$P\left(\Delta(\mathcal{T}_n) > \log^3(n) n^{1/2} \mid \text{Structure of } \mathcal{T}_n\right) \leq |\mathcal{T}_n| P\left(\sup_{0 \leq k \leq \varnothing(\mathcal{T}_n)} |Z_k| \geq \log^3(n) n^{1/2}\right).$$

On the event $A_n := \{|\mathcal{T}_n| \leq n^3 \text{ and } \varnothing(\mathcal{T}_n) \leq n \log^2(n)\}$ the right-hand side of the last display is $O(n^{-2})$. But the previous estimates imply that A_n eventually occur and thus an application of Borel-Cantelli proves $\Delta(\mathcal{T}_n) \preceq n^{1/2}$. \square

2.2 Schaeffer construction

A rooted quadrangulation is associated with (T_∞, ℓ) by the following device. We first embed the labeled tree T_∞ in the plane such that there is no accumulation point for the vertices and such that the edges are not crossing (this is possible since T_∞ has one spine). Then for each corner c of T_∞ , we draw an edge between c and its successor $\mathcal{S}(c)$ (note that this successor exists a.s.). All the edges can be drawn in a non-crossing fashion and after erasing the (embedding of the) tree, the resulting map is an infinite quadrangulation. See Fig. 4.

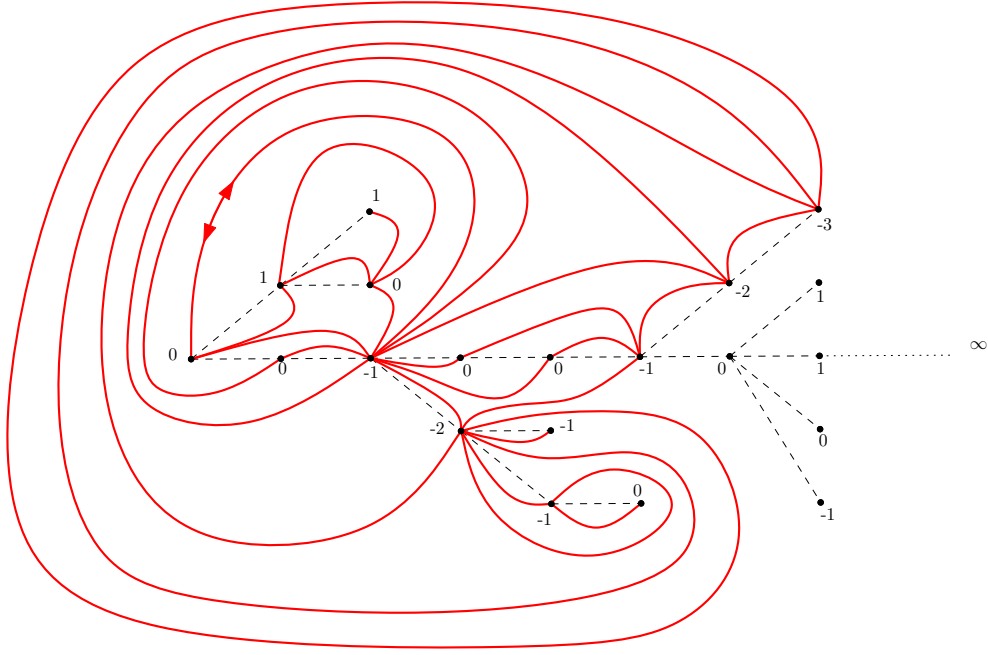


Figure 4: Illustration of the construction of $\Phi(T_\infty, \ell)$. The orientation of the root edge is given by an extra Bernoulli variable.

We root it at the edge emanating from the root corner of T_∞ whose orientation is given by an extra independent Bernoulli variable $\eta \in \{+, -\}$. The quadrangulation that we obtain, denoted by $\Phi(T_\infty, \ell)$ (the dependance in η is implicit), has the same distribution as Q_∞ , see [18]. In this representation, the vertices of the map are exactly the vertices of the tree T_∞ , and we shall always make this identification. Using the fact that any neighboring vertices in $\Phi(T_\infty, \ell)$ must have labels that differ by 1 in absolute value, we easily get that for every $u, v \in T_\infty$ we have

$$d_{\text{gr}}(u, v) \geq |\ell(u) - \ell(v)|. \quad (10)$$

In fact, the labeling ℓ of the vertices of $\Phi(T_\infty, \ell)$ inherited from this construction has a metric meaning within the quadrangulation $\Phi(T_\infty, \ell)$: The main result of [18] states that for every $u, v \in \Phi(T_\infty, \ell)$ we have

$$\ell(u) - \ell(v) = \lim_{z \rightarrow \infty} (d_{\text{gr}}(z, u) - d_{\text{gr}}(z, v)),$$

where $z \rightarrow \infty$ means that $d_{\text{gr}}(\rho, z) \rightarrow \infty$. We will not use this precise result in what follows, however we will make a great use of the following bounds on the distances in $\Phi(T_\infty, \ell)$. First of all, the very standard bound

$$d_{\text{gr}}(u, v) \leq 2 + \ell(u) + \ell(v) - 2 \max \left\{ \min_{[c, c']} \ell : \{\mathcal{V}(c), \mathcal{V}(c')\} = \{u, v\} \right\}, \quad (11)$$

which can be proved as follows. Consider a corner c_i of u and a corner c_j of v and suppose that $i \leq j$. We construct the path starting from c_i and c_j following iteratively their successors. These two paths merge at the first corner after c_j with label $\min_{[c_i, c_j]} \ell - 1$ and the concatenation of these two paths up to the merging point gives the bound $d_{\text{gr}}(u, v) \leq 2 + \ell(u) + \ell(v) - 2 \min\{\ell(c) : c \in [c_i, c_j]\}$. The other cases are similar. We also have a lower bound also called *cactus bound*

$$d_{\text{gr}}(u, v) \geq \ell(u) + \ell(v) - 2 \min \{ \ell(w) : w \in \llbracket u, v \rrbracket \}, \quad (12)$$

where $\llbracket u, v \rrbracket$ is the geodesic line between u and v in the tree T_∞ . Let us sketch the idea of the proof of this lower bound, see [18, Equation (4)]. Excluding trivial cases, we consider a vertex $w \in \llbracket u, v \rrbracket \setminus \{u, v\}$ such that $\ell(w) < \ell(u)$ and $\ell(w) < \ell(v)$. Then choose two corners c, c' of w on both sides of $\llbracket u, v \rrbracket$. Here also we consider the two paths formed by the successors of c and c' . These two paths merge and their concatenation forms a loop separating u from v in $\Phi(T_\infty, \ell)$. Thus by Jordan's lemma any path going from u to v in Q_∞ must encounter this loop. To finish notice that all the labels on the loop are less than or equal to the label of w and use the bound (10) to conclude. We safely leave the details to the reader.

3 Geometric estimates

In the following, we will consider that the UIPQ is constructed from a uniform infinite labeled tree, that is, we set $Q_\infty = \Phi(T_\infty, \ell)$. We shall always identify the vertices of Q_∞ with those of T_∞ . Unless mentioned, $d_{\text{gr}}(\cdot, \cdot)$ stands for the graph distance in Q_∞ .

3.1 Uniform estimates on the degrees

Our first geometric matter concerns the degrees of the vertices in Q_∞ . Angel & Schramm proved that the degree of the origin of the UIPT has an exponential tail, see [5, Lemma 4.1]. We shall provide the exact distribution of the degree of the origin of the UIPQ and give a uniform control among all vertices within a given distance from the origin ρ of Q_∞ .

Proposition 9. (i) *For every $y \in (0, 6/5)$, we have*

$$E \left[y^{\deg(\rho, Q_\infty)} \right] = \frac{y}{12} (1 + y/2)^{-1/2} (1 - 5y/6)^{-3/2}.$$

In particular, $P(\deg(\rho, Q_\infty) = k) \sim \sqrt{k/40\pi} (5/6)^k$ as $k \rightarrow \infty$.

(ii) *Furthermore if D_r denotes the maximal degree of a vertex in $\text{Ball}(Q_\infty, r)$, then there exists a constant $K_1 > 0$ such that, almost surely*

$$\limsup_{r \rightarrow \infty} \frac{D_r}{\log(r)} \leq K_1.$$

Proof of Proposition 9 part (i). This result follows from the enumeration of general planar maps. Indeed, there is a well-known bijection \mathcal{D} between the set of all rooted planar maps with n edges and the set of all quadrangulations with n faces. The application \mathcal{D} can be described as follows: If m is a planar map with n edges, then in each face of m we put an extra point that we link to all the vertices adjacent to this face. We then erase all the edges of m and are left with a quadrangulation q with n faces, see Fig. 5. The root edge of m is the first edge on the right of the

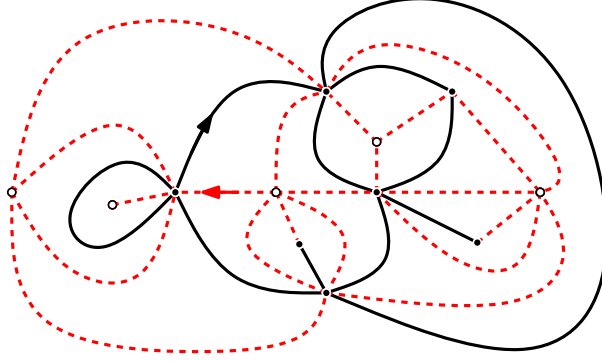


Figure 5: Illustration of the duality between maps and quadrangulations.

rooted edge of q as depicted on Fig. 5. In this correspondence, the degree of the root face (on the right of the root edge) of m is equal to the degree of the origin of the root edge of q . Part (i) then directly follows from (1) and Theorem 1 of [24]. \square

Before going into the proof of Proposition 9 part (ii) let us give a lemma on T_∞ . Recall that the contour of T_∞ is denoted by $(\dots, c_{-2}, c_{-1}, c_0, c_1, c_2, \dots)$ and its spine by (s_0, s_1, s_2, \dots) . Assume that T_∞ has been drawn in the plane and consider the sequence of oriented edges $(\dots, \vec{e}_{-2}, \vec{e}_{-1}, \vec{e}_0, \vec{e}_1, \vec{e}_2, \dots)$ obtained when doing the contour of the tree in clockwise order such that \vec{e}_0 is the first oriented edge encountered after the root corner of the tree T_∞ . Since T_∞ almost surely has one spine, for any oriented edge \vec{e} of T_∞ one can say if \vec{e} is pointing towards or from infinity, formally if \vec{e}_- and \vec{e}_+ are the origin and endpoints of \vec{e} the following quantity is well-defined

$$\zeta(\vec{e}) := \lim_{z \rightarrow \infty} (\text{dist}(z, \vec{e}_+) - \text{dist}(z, \vec{e}_-)) \in \{-1, +1\},$$

where $\text{dist}(\cdot, \cdot)$ is the graph distance in T_∞ and $z \rightarrow \infty$ means that $\text{dist}(z, \emptyset) \rightarrow \infty$.

Lemma 10. *If T_∞ is a critical geometric Galton-Watson tree conditioned to survive then the variables $\{\zeta(\vec{e}_i), i \in \mathbb{Z}\}$ are i.i.d. Bernoulli variables of parameter $1/2$.*

This lemma easily follows from [18, Lemma 4] or [32]. We leave to the reader the fact that this lemma together with $T_\infty \in \mathcal{S}$ completely characterizes the distribution of T_∞ . In particular, we deduce that for any $k \in \mathbb{Z}$, the tree $T_\infty^{(k)}$ consisting of T_∞ re-rooted at the corner c_k (and the same planar ordering) has the same distribution as T_∞ ,

$$T_\infty^{(k)} \stackrel{(d)}{=} T_\infty. \tag{13}$$

Proof of Proposition 9 (ii). By part (i), the degree of ρ in Q_∞ has an exponential tail. Since $\rho = \emptyset$ with probability $1/2$ we deduce that the degree of \emptyset in Q_∞ has an exponential tail as well. By invariance of T_∞ under re-rooting, we deduce that there exists some constant $c > 0$ such that $P(\deg(\mathcal{V}(c_r)) \geq c \log(r)) = O(r^{-2})$, where $\deg(\cdot)$ denotes the degree in Q_∞ . Applying Borel-Cantelli's lemma we deduce that a.s. we eventually have

$$\deg(\mathcal{V}(c_i)) \leq c \log(r) \quad \text{for all } |i| \leq r. \quad (14)$$

For $r \geq 1$, let σ_r be the first $i \geq 0$ such that the i^{th} vertex along the spine of T_∞ has label $\ell(s_i) = -r$. Recall that the tree T_∞ pruned at s_{σ_r} is denoted by \mathcal{T}_{σ_r} . Thanks to (12) we deduce that if $v \in Q_\infty$ is such that $d_{\text{gr}}(\emptyset, v) \leq r - 1$ then $v \in \mathcal{T}_{\sigma_r}$. Since the graph distance in Q_∞ between \emptyset and the origin ρ of Q_∞ is either 0 or 1 we deduce that

$$\begin{aligned} \text{Ball}(Q_\infty, r) &\subset \{u \in Q_\infty : d_{\text{gr}}(\rho, u) \leq r + 1\} \\ &\subset \{u \in Q_\infty : d_{\text{gr}}(\emptyset, u) \leq r + 2\} \\ &\subset \mathcal{T}_{\sigma_r+3}, \end{aligned} \quad (15)$$

in terms of vertex sets. If I_r and S_r are the minimal resp. maximal indices of a corner belonging to \mathcal{T}_{σ_r} then arguments similar to that of the proof of Proposition 8 show that $S_r - I_r \approx r^4$ (in fact $S_r - I_r \preceq r^\alpha$ for some $\alpha > 0$ would suffice here). Using this and (14), we deduce that there exists a constant K_1 such that a.s. for every $I_r \leq i \leq S_r$ we have $\deg(\mathcal{V}(c_i), Q_\infty) \leq K_1 \log(r)$. Using (15), we complete the proof of the proposition. \square

In particular, we deduce from (6) and the last proposition that

$$\#\{\text{pioneer times} \leq \tau_n\} \approx n. \quad (16)$$

3.2 Growth

We establish the analogs of the theorem of Angel [3] about the volume growth of the UIPT in the case of the UIPQ. Recall that $\text{Ball}(Q_\infty, r)$ is composed of the faces of Q_∞ that have at least one vertex at distance strictly less than r from the origin $\rho \in Q_\infty$, and that $|\text{Ball}(Q_\infty, r)|$ is the number of vertices of $\text{Ball}(Q_\infty, r)$.

Proposition 11. *We have $|\text{Ball}(Q_\infty, r)| \approx r^4$.*

See also [16, 33] for closely related results.

Proof. Here also we consider that $Q_\infty = \Phi(T_\infty, \ell)$. We begin with the upper bound $|\text{Ball}(Q_\infty, r)| \preceq r^4$. Let $r \geq 1$. As in the proof of Proposition 9 we use the tree \mathcal{T}_{σ_r} consisting of T_∞ pruned at the first vertex s_{σ_r} of the spine reaching label $-r$. Recall (15). Since σ_r is the hitting time of $-r$ by a random walk with steps distribution uniform in $\{-1, 0, +1\}$, we have $\sigma_r = \sigma_1^{(1)} + \dots + \sigma_1^{(r)}$ where the $\sigma_1^{(i)}$ are i.i.d. and distributed as σ_1 . Standard calculations show that $P(\sigma_1 \geq n) \sim Cn^{-1/2}$ for some $C > 0$. Hence similar arguments as those presented in the proof of Proposition 8 show that $\sigma_r \approx r^2$. We can thus combine this fact together with (8) and (15) to complete the upper bound.

We now turn to the lower bound. For $r \geq 1$, we put

$$L_r = \sup\{i \geq 0 : \Delta(\mathcal{T}_i) < r\}.$$

Consistently with the preceding notation we write \mathcal{T}_{L_r} for the tree T_∞ pruned at s_{L_r} . Using the bound (11), one sees that all the vertices in \mathcal{T}_{L_r} are at a graph distance at most $3r + 2$ from \emptyset in Q_∞ , which implies

$$\mathcal{T}_{L_r} \subset \text{Ball}(Q_\infty, 3r + 4), \quad (17)$$

in terms of vertex sets. Using (9) we deduce that $L_r \approx r^2$. Henceforth by (8) we have $|\mathcal{T}_{L_r}| \approx r^4$ which together with (17) completes the proof of the proposition. \square

3.3 Tentacles

Our third estimate deals with the distances in the hull of the ball of radius r in Q_∞ . Recall that $\overline{\text{Ball}}(Q_\infty, r)$ is obtained from $\text{Ball}(Q_\infty, r)$ after filling-in all the finite holes. We show that in fact this procedure does not increase the diameter by much, that is, $\overline{\text{Ball}}(Q_\infty, r)$ does not grow long “tentacle”.

Proposition 12. *We have $\max\{d_{\text{gr}}(\rho, u) : u \in \overline{\text{Ball}}(Q_\infty, r)\} \approx r$.*

Proof. We use the same notation as in the proof of Proposition 11. Note that the lower bound $\max\{d_{\text{gr}}(\rho, u) : u \in \overline{\text{Ball}}(Q_\infty, r)\} \succeq r$ is trivial. For the upper bound, we will strengthen (15) and prove that

$$\overline{\text{Ball}}(Q_\infty, r) \subset \mathcal{T}_{\sigma_{r+3}}, \quad (18)$$

in terms of vertex set in Q_∞ . Indeed consider $s_{\sigma_{r+3}}$ the first vertex on the spine of T_∞ with label $-r - 3$ and pick c and c' two corners associated with $s_{\sigma_{r+3}}$ from both sides of the spine. We then draw the two paths in Q_∞ starting from c and c' by following the chain of successors. These two paths eventually merge. We consider the cycle \mathcal{C} formed by the two paths up to the merging point. It is composed of vertices of labels less than $-r - 3$ and thus by (10) at distance at least $r + 3$ from \emptyset . Since \mathcal{C} separates Q_∞ into two parts and because the minimal graph distance from a point on the cycle to the origin ρ is at least $r + 2$, we deduce that $\overline{\text{Ball}}(Q_\infty, r)$ is contained in the finite part of $Q_\infty \setminus \mathcal{C}$ which is included in $\mathcal{T}_{\sigma_{r+3}}$ in terms of vertex sets. It follows from (11) that

$$\max\{d_{\text{gr}}(\rho, u) : u \in \overline{\text{Ball}}(Q_\infty, r)\} \leq 2 + 3\Delta(\mathcal{T}_{\sigma_{r+3}}).$$

The proof is completed by using (9) and the fact that $\sigma_r \approx r^2$. \square

Remark. *As a corollary of Proposition 11 and 12 we have $|\overline{\text{Ball}}(Q_\infty, r)| \approx r^4$.*

3.4 Separating cycles

Recall the notation γ_r for the separating cycle “at distance $r \geq 0$ ” from the origin in Q_∞ and $|\gamma_r|$ for its length. Krikun [28] showed that a slight variant of $|\gamma_r|$ is approximately of order r^2 and that once renormalized by r^2 it converges in distribution towards a $\Gamma(3/2)$ law. Here we use his results to show:

Proposition 13. *We have $|\gamma_r| \approx r^2$.*

Proof. In [28], Krikun studied a separating cycle closely related to our γ_r . More precisely he considered the cycle $\tilde{\gamma}_r$ formed by the vertices at distance r from ρ and the diagonals of the faces of Q_∞ between them such that $\tilde{\gamma}_r$ separates ρ from the infinite part of the quadrangulation. Since γ_r and $\tilde{\gamma}_r$ are within distance 2 from each other, by Proposition 9 we have $|\gamma_r| \approx |\tilde{\gamma}_r|$ and it thus suffices to prove $|\tilde{\gamma}_r| \approx r^2$. Krikun explicitly computed the transition probabilities of $|\tilde{\gamma}_r|$ and showed that the process $(|\tilde{\gamma}_r|)_{r \geq 1}$ is a time-reversed critical branching process with offspring distribution in the domain of attraction of a stable distribution of parameter $3/2$. More precisely we have (Theorem 2 in [28])

$$P\left(|\tilde{\gamma}_{n+r}| = k \mid |\tilde{\gamma}_r| = l\right) = \frac{[t^k]F(t)}{[t^l]F(t)} P(\xi_n = l \mid \xi_0 = k), \quad (19)$$

where ξ is a critical branching process with an explicit offspring distribution and $F(t) = 3/4(\sqrt{(9-t)/(1-t)}-3)$ is the generating function of its stationary measure. In particular, we have [28, Proof of Corollary 1]

$$P(|\tilde{\gamma}_r| = m) \leq C m^{1/2} r^{-3} \left(1 - \frac{2}{r^2}\right)^m, \quad (20)$$

for some constant $C > 0$ (uniform in m, r). We immediately deduce that

$$P(|\tilde{\gamma}_r| \geq r^2 \log(r)) \leq C r^{-3} \sum_{k \geq r^2 \log(r)} k^{1/2} \left(1 - \frac{2}{r^2}\right)^k.$$

Regrouping the terms in the right hand side by packets of r^2 we get for large $r \geq 0$

$$P(|\tilde{\gamma}_r| \geq r^2 \log(r)) \leq C' \sum_{k \geq \log(r)} k^{1/2} e^{-2k} = O(r^{-2+\varepsilon}).$$

A direct application of Borel-Cantelli's lemma shows that $|\tilde{\gamma}_r| \leq \log(r)r^2$ eventually which proves the upper bound of the proposition. The lower bound is a bit more involved. First of all, it follows from (20) that $P(|\tilde{\gamma}_r| \leq r^2 \log^{-1}(r)) = O(\log^{-3/2}(r))$. Applying Borel-Cantelli's lemma we get that eventually $|\tilde{\gamma}_r| \geq r^2 \log^{-1}(r)$ along the values of $r = 2, 4, 8, \dots, 2^k, \dots$. Notice however that the random process $(|\tilde{\gamma}_r|)_{r \geq 1}$ is *not* increasing and thus we cannot interpolate between values of 2^k . We bypass this problem by using the Markovian nature of $(\tilde{\gamma})$.

To simplify notation we set $l(r) = \lfloor r^2 \log^{-1}(r) \rfloor$ and $u(r) = \lceil r^2 \log(r) \rceil$. We just proved that a.s. we eventually have

$$l(2^k) \leq |\tilde{\gamma}_{2^k}| \leq u(2^k). \quad (21)$$

For $r \geq 1$ we let A_r be the following event

$$A_r := \left\{ \begin{array}{l} l(r) \leq |\tilde{\gamma}_r| \leq u(r), \\ |\tilde{\gamma}_i| \leq i^2 \log^{-10}(i), \text{ for some } r \leq i \leq 2r, \\ l(2r) \leq |\tilde{\gamma}_{2r}| \leq u(2r). \end{array} \right\}.$$

We claim that $P(A_r) = O(\log^{-2}(r))$. This is sufficient to finish the proof of Proposition 13: By applying Borel-Cantelli's lemma to the sequence of events A_{2^k} for $k = 1, 2, 3, \dots$ we deduce that A_r^c eventually holds which combined with (21) yields to $|\tilde{\gamma}_r| \approx r^2$. Let us now prove the claim. By definition, $P(A_r)$ is equal to

$$\begin{aligned} &= \sum_{a=l(r)}^{u(r)} \sum_{b=l(2r)}^{u(2r)} P(|\tilde{\gamma}_r| = a) P\left(\exists r \leq i \leq 2r : |\tilde{\gamma}_i| \leq r^2 \log^{-10}(r) \text{ and } |\tilde{\gamma}_{2r}| = b \mid |\tilde{\gamma}_r| = a\right), \\ &= \sum_{a=l(r)}^{u(r)} \sum_{b=l(2r)}^{u(2r)} \frac{[t^b]F(t)}{[t^a]F(t)} P(|\tilde{\gamma}_r| = a) P\left(\exists 0 \leq i \leq r : \xi_i \leq r^2 \log^{-10}(r) \text{ and } \xi_r = a \mid \xi_0 = b\right), \end{aligned}$$

by (19). Using standard singularity analysis one shows that $[t^m]F(t) \sim 3/\sqrt{2\pi m}^{-1/2}$ as $m \rightarrow \infty$. Using this and (20) we can bound the term $[t^b]F(t)/[t^a]F(t)P(|\tilde{\gamma}_r| = a)$ in the last display by $Cr^{-2} \log^{3/2}(r)$ for some constant $C > 0$ uniform in $r \geq 1$. Thus we have

$$P(A_r) \leq Cr^{-2} \log^{3/2}(r) \sum_{b=l(2r)}^{\lfloor u(2r) \rfloor} P(\exists 0 \leq i \leq r : \xi_i \leq r^2 \log^{-10}(r), l(r) \leq \xi_r \leq u(r) \mid \xi_0 = b).$$

Fix $l(2r) \leq b \leq u(2r)$ and let us estimate the probability that the branching process ξ starting from $\xi_0 = b$ reaches a level lower than $r^2 \log^{-10}(r)$ for some $0 \leq i \leq r$ and finally ends at a state $l(r) \leq \xi_r$. Since ξ is a critical branching process, it is in particular a martingale. Thus if we introduce the stopping times $T_r = \inf\{n \geq 0 : \xi_n \geq l(r)\}$ and $\tau = \inf\{n \geq 0 : \xi_n = 0\}$ we deduce that

$$P(T_r < \tau < r \mid \xi_0 = i) \leq \frac{i}{l(r)}.$$

As a consequence, applying the Markov property of ξ at the first time j where $\xi_j \leq r^2 \log^{-10}(r)$ we deduce that the event $\{\xi_0 = b \rightarrow \xi_j \leq r^2 \log^{-10}(r) \rightarrow \xi_r \geq l(r)\}$ for the branching process ξ has a probability less than or equal to $r^2 \log^{-10}(r)/l(r) \sim \log^{-9}(r)$. Gathering-up the pieces we finally get that $P(A_r) = O(\log^{-2}(r))$ as desired. \square

3.5 Aperture after peeling

Let $Q_0 \subset Q_1 \subset Q_2 \subset \dots$ be the sequence of quadrangulations with boundary obtained by a peeling of Q_∞ . For each $n \geq 0$ we know that $Q_\infty \setminus Q_n$ has the same distribution as a UIPQ of the $|\partial Q_n|$ -gon. Following [19], if q is a quadrangulation with a boundary, we denote the maximal distance between any pair of points of ∂q by $\text{aper}(q)$ and call it *the aperture* of q .

Proposition 14. *We have $\text{aper}(Q_\infty \setminus Q_n) \preceq |\partial Q_n|^{1/2}$.*

We slightly abuse notation in the last proposition. Of course the reader would have understood that $\text{aper}(Q_\infty \setminus Q_n) \preceq |\partial Q_n|^{1/2}$ means that there exists a constant $\kappa > 0$ such that almost surely we eventually have $\text{aper}(Q_\infty \setminus Q_n) \leq \log^\kappa(n) |\partial Q_n|^{1/2}$.

Proof. We already recalled that for every $n \geq 0$ the quadrangulation with a simple boundary $Q_\infty \setminus Q_n$ has the same distribution as a UIPQ of the $|\partial Q_n|$ -gon. We now recall an estimate of [19]:

Theorem ([19]). *There exists $c, c' > 0$ such that for all $p \geq 1$ and $\lambda > 0$ the aperture of a uniform infinite planar quadrangulation with simple boundary of perimeter $2p$ satisfies*

$$P(\text{aper}(Q_{\infty, 2p}) \geq \lambda \sqrt{p}) \leq c p^{2/3} \exp(-c' \lambda^{2/3}).$$

Thus taking $\lambda = \log^3(n)$ in this theorem and noticing that $|\partial Q_n|$ is deterministically less than $2n$, an application of Borel-Cantelli's lemma finishes the proof. \square

4 Remaining proofs

We begin with the proof of Theorem 5.

4.1 Peeling estimate

Proof of Theorem 5. Because of Lemma 4 it is sufficient to prove Theorem 5 for *one* peeling algorithm. We thus consider Q_0, Q_1, \dots the peeling of Q_∞ using Algorithm \mathcal{L} of Section 1.3. During this peeling we know from Proposition 6 that all the edges on the separating cycles $\{\gamma_r, r \geq 0\}$ must be part of the boundary of some Q_n . Furthermore, for all $n \geq 0$, any edge on the boundary of Q_n must be at a graph distance less than 2 from a separating cycle γ_r for some $r \geq 0$. Using the uniform estimates on the degree (Proposition 9) and Proposition 6 (and the remark after it) we deduce that after n steps of peeling, the boundary ∂Q_n of Q_n is located at a graph distance less than 2 from some γ_{R_n} with

$$\sum_{i=1}^{R_n} |\gamma_i| \approx n.$$

Coupling the last display with the fact that $|\gamma_i| \approx i^2$ (Proposition 13) we deduce that $R_n \approx n^{1/3}$.

Using this with Proposition 13 and 9 again, we deduce that since $d_{\text{gr}}(\partial Q_n, \gamma_{R_n}) \leq 2$ we have $|\partial Q_n| \approx n^{2/3}$. This proves the first half of Theorem 5.

Let us now focus on the volume of Q_n . From the deductions made above we have

$$|\overline{\text{Ball}}(Q_\infty, R_n - 3)| \leq |Q_n| \leq |\overline{\text{Ball}}(Q_\infty, R_n + 3)|.$$

We then use $R_n \approx n^{1/3}$ and the remark after Proposition 12 to get $|Q_n| \approx n^{4/3}$. This completes the proof of Theorem 5. \square

4.2 Pioneer points and subdiffusivity

With all the estimates that we now have in our hands, the proof of Theorem 1 is effortless.

Proof of Theorem 1. Let Q_∞ be the uniform infinite planar quadrangulation and conditionally on Q_∞ , let $(X_n)_{n \geq 0}$ be a nearest-neighbor simple random walk starting from the origin $\rho \in Q_\infty$. We consider Q_0, Q_1, \dots the peeling Q_∞ according to Algorithm \mathcal{W} of Section 1.4. By Theorem 5 we have $|\partial Q_n| \approx n^{2/3}$ and applying Proposition 14 we deduce that $\text{aper}(Q_\infty \setminus Q_n) \preceq n^{1/3}$. Let D_n^- and D_n^+ be the minimal and maximal distance to the origin $\rho \in Q_\infty$ of a vertex in ∂Q_n . Since we have $\text{aper}(Q_\infty \setminus Q_n) \geq D_n^+ - D_n^-$ we deduce that $D_n^+ - D_n^- \preceq n^{1/3}$. From the inclusions

$$\overline{\text{Ball}}(Q_\infty, D_n^- - 1) \subset Q_n \subset \overline{\text{Ball}}(Q_\infty, D_n^+ + 1),$$

and since $|Q_n| \approx n^{4/3}$ (by Theorem 5) we deduce using Proposition 11 and 12 that $D_n^- \preceq n^{1/3}$ and $D_n^+ \succeq n^{1/3}$. But since $D_n^+ - D_n^- \preceq n^{1/3}$ we must have $D_n^+ \preceq n^{1/3}$. To finish the proof, just recall that Q_n is the quadrangulation discovered when n steps of peeling have been demanded and that during this time the SRW has discovered $\approx n$ pioneer points by (16). Hence the pioneer points discovered so far are contained in $\overline{\text{Ball}}(Q_\infty, D_n^+ + 1)$ which has a diameter $\approx n^{1/3}$ by Proposition 12. Finally, at least one pioneer points is at distance at least $D_n^+ - 2$ from ρ , hence

$$\max_{1 \leq i \leq n} d_{\text{gr}}(\rho, P_i) \approx n^{1/3}.$$

□

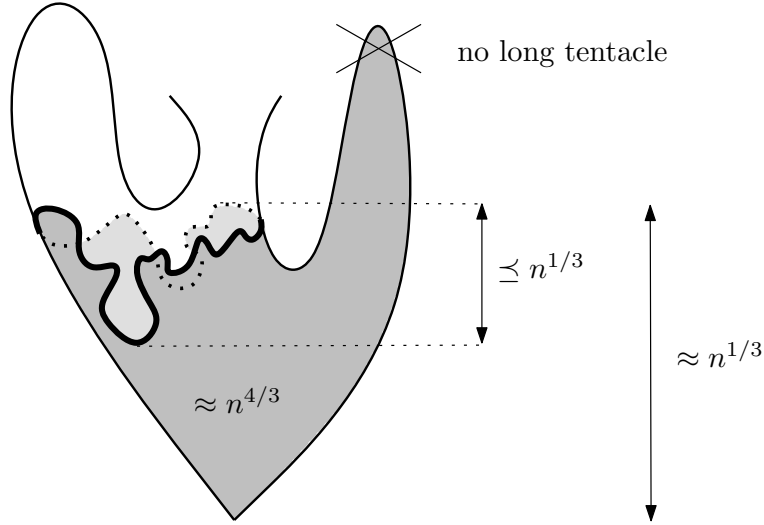


Figure 6: Illustration of the proof of Theorem 1. The curvy line represents the boundary of Q_n .

Proof of Corollary 2. With the notation of the proof of Theorem 1 the range $\{X_1, X_2, \dots, X_{\tau_n}\}$ is contained in $\overline{\text{Ball}}(Q_\infty, D_n^+ + 1)$. We obviously have $\tau_n \succeq n$ (and in fact τ_n could be much larger than n). We then use $D_n^+ \approx n^{1/3}$ and Proposition 12 to conclude. □

Remark. *It is clear from the proof of Corollary 2 that the $1/3$ exponent of subdiffusivity is not likely to be sharp. Indeed, most of the times are not pioneer times for the simple random walk since two pioneer times could be separated by a long period of time. Yet this phenomenon is hard to control.*

5 Comments and questions

Before making a more precise list of comments, let us emphase the fact that we focused on the UIPQ for sake of simplicity and because many tools are already available for this model. There should not be major conceptual problem in generalizing our result to other type of random lattices such as the UIPT or Boltzmann maps – but the required technics might be (much!) more difficult to work with.

5.1 Peeling

The proof of Theorem 1 is not specific to the peeling with algorithm \mathcal{W} and can be generalized to show that for any peeling $Q_0 \subset Q_1 \subset Q_2 \subset \dots$ of Q_∞ we actually have

$$\max_{u \in Q_n} d_{\text{gr}}(\rho, u) \approx n^{1/3}.$$

In particular, this result can be applied with other peeling procedures among which:

- The peeling along layers of Q_∞ (giving back a few estimates of Section 3),
- The peeling along a percolation interface as developed in [4],
- The peeling associated with internal diffusion limited aggregation on the UIPQ,
- The peeling along a Brownian motion on the Riemann surface associated with Q_∞ , see [25],
- ...

Limit processes. In Theorem 5 we established the rough estimates $|\partial Q_n| \approx n^{2/3}$ and $|Q_n| \approx n^{4/3}$. One can ask for a precise limit theorem of the re-normalized processes

$$(n^{-2/3}|\partial Q_{\lfloor nt \rfloor}|, n^{-4/3}|Q_{\lfloor nt \rfloor}|)_{t \geq 0}.$$

Note that the two components are not independent and that Angel [3] conjectured that (in the triangulation case) the first component converges towards a stable process of parameter $3/2$ conditioned to remain positive see [12].

Greedy peeling. Another useful property that has to be addressed about the peeling process is the following. For any peeling $Q_0 \subset Q_1 \subset Q_2 \subset \dots$ of Q_∞ show that we have

$$\bigcup_{n \geq 0} Q_n = Q_\infty.$$

In words, whatever the algorithm used to peel Q_∞ , we eventually discover the whole quadrangulation Q_∞ . This would be implied by the fact that during the peeling

there exist infinitely many times such that $|\partial Q_{n+1}| \leq |\partial Q_n|/2$. This result would have nice applications: Applying it with Algorithm \mathcal{W} it should imply that the range of a simple random walk $(X_n)_{n \geq 0}$ creates infinitely many loops separating the origin $\rho \in Q_\infty$ from ∞ a.s.. In particular, two independent simple random walk paths on Q_∞ would intersect showing that Q_∞ is almost surely Liouville, see [9]. We expect a similar result to hold for the range of Brownian motions on the Riemann surface of Q_∞ thus yielding a different perspective on the result of [25]. We hope to pursue these goals in future works.

5.2 Sudiffusivity

Note that our subdiffusivity result (Corollary 2) was not based on resistance nor heat kernel estimates as it is generally the case. In reward we can give bounds on the probability that a simple random walk returns to the origin in n steps. For any x, y in Q_∞ and $n \geq 0$, we denote by $p(x, y, n)$ the probability that a SRW started at x hits the point y at time n . Note that $p(x, y, n)$ is random. Our main result implies,

Corollary 15. *We have $p(\rho, \rho, 2n) \succeq n^{-4/3}$.*

Proof. Recall that D_r denotes the maximal degree within distance r of ρ . For any $r \geq 1$ we have

$$\begin{aligned} p(\rho, \rho, 2n) &\geq \sum_{x \in \text{Ball}(Q_\infty, r)} p(\rho, x, n) p(x, \rho, n) \\ &= \sum_{x \in \text{Ball}(Q_\infty, r)} \frac{\deg(\rho)}{\deg(x)} p(\rho, x, n)^2 \\ &\geq D_r^{-1} \left(\sum_{x \in \text{Ball}(Q_\infty, r)} p(\rho, x, n) \right)^2 |\text{Ball}(Q_\infty, r)|^{-1} \\ &\geq \frac{P(X_n \in \text{Ball}(Q_\infty, r))^2}{D_r |\text{Ball}(Q_\infty, r)|}, \end{aligned} \tag{22}$$

where we used Cauchy-Schwarz inequality to go from the second to the third line. Taking $r = \lfloor n^{1/3} \log^\kappa(n) \rfloor$ for some $\kappa > 0$ we deduce from Proposition 9, Proposition 11 and Corollary 2 that $p(\rho, \rho, 2n)$ is asymptotically larger than $n^{-4/3} \log^{\kappa'}(n)$ for some $\kappa' > 0$. This completes the proof of the corollary. \square

Remark. Notice that we can produce a lower bound on the displacement of the SRW on the UIPQ by using the crude fact that the electrical resistance $R_{x,y}$ between two points $x, y \in Q_\infty$ is less than or equal to $d_{\text{gr}}(x, y)$, see also [6]. Let us for example given an upper bound on the mean of

$$E_r = \inf\{n \geq 0 : X_n \notin \text{Ball}(Q_\infty, r)\}.$$

By the result of [15] we have

$$E[T_r] \leq 2|\text{Ball}(Q_\infty, r)| R_{\rho, \gamma_r} \leq r^{5+o(1)}.$$

We do not sharpen this result because we do not believe that this is the right exponent.

The subdiffusive behavior of the SRW on Q_∞ established in Corollary 2 is not sufficient to conclude recurrence of the walk. Still, we believe that Q_∞ is recurrent (see Conjecture 2) and that the subdiffusivity exponent is critical for deciding recurrence or transience, see Conjecture 1.

We also suspect that one does not need the detailed structure of the UIPQ to establish subdiffusivity but only the existence of bottlenecks at all scales. In particular, is it the case that any planar stationary random graph (see [9]) with volume growth bigger than quadratic is subdiffusive for the simple random walk? See related conjectures in [10].

5.3 KPZ

This part is heuristic. For a mathematically precise statement of the KPZ relations, the reader should consult [23].

Verification of KPZ relation for pioneer exponents. The famous KPZ relation [27] predicts that certain exponents of statistical mechanics models on a random planar map are related to the analogous exponents on a regular lattice, see [21]. More precisely, let F be a random fractal on a Euclidean space (for example the set of pioneer points of a Brownian motion). If F has “dimension” $2(1-x)$ that means, roughly speaking, that $\varepsilon^{-2(1-x)}$ balls of radius ε are necessary to cover F when $\varepsilon \rightarrow 0$. Then x is called the *Euclidean scaling exponent of F* [23]. Similarly, if we consider the same random fractal on a random geometry one can define its “quantum scaling exponent” to be Δ if the number of balls of radius ε (in the random geometry) needed to cover F is approximatively $(n_\varepsilon)^{(1-\Delta)}$ where n_ε is the number of balls needed to cover the full space. The KPZ relation then predicts

$$x = \frac{\gamma^2}{4}\Delta^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta, \quad (\text{KPZ})$$

where $0 \leq \gamma < 2$ is a parameter depending on the features of the model that produced the random fractal. In particular, in the case of fractals coming from a Brownian motion we should have $\gamma = \sqrt{8/3}$.

Going to a discrete level, a random subset \mathcal{F}_n of a planar quadrangulation with n faces is said to have a quantum scaling exponent Δ_D if $|\mathcal{F}_n|$ is of order $n^{1-\Delta_D}$ as $n \rightarrow \infty$. Taking a ball of radius r in the UIPQ, we know by Theorem 1 that $\approx r^3$ pioneer points are visited before the walk exits this ball which contains $\approx r^4$ points. Putting this together, we deduce that the discrete quantum scaling exponent for pioneer points is $\Delta_D = 1/4$. Going through (KPZ) this becomes $x_D = 1/8$. Indeed, $2 - 2x_D = 7/4$ is the dimension of the set of pioneer points of the Brownian motion as identified by Lawler, Schramm, Werner [30] in the Euclidean case. Notice that various quantum scaling exponent for simple random walk on random lattices were derived non-rigorously by Duplantier & Kwon [22].

Support for Conjecture 1. Let us use once more the KPZ relation for intersection exponents of simple random walks. More precisely, the probability that $L \geq 1$ independent random walks starting from the same point in a random lattice are not intersecting each other up to time n is supposed to decay as $n^{-\Delta_L + o(1)}$.

These exponents can be derived from the Euclidean case [30] using the KPZ relation and we have [21, Eq (3.14)]

$$\Delta_L = \frac{1}{2} \left(L - \frac{1}{2} \right).$$

The special case $\Delta_1 = 1/4$ corresponds to the disconnection exponent, meaning that the probability that the origin of *one* walk has not been disconnected from infinity after n steps decays as $n^{-1/4+o(1)}$. By time reversing this propability is also that of the n^{th} step of the walk being a pioneer point. Thus we should have

$$P(n \text{ is a pioneer time}) \asymp n^{-1/4+o(1)}.$$

Henceforth, in n^4 steps the SRW should have discovered roughly $\sum_{k=0}^{n^4} k^{-1/4} \asymp n^3$ pioneer points and by Theorem 1 the maximal displacement from the root in the first n^4 steps is $\approx n$. This supports Conjecture 1. Note that this is also equivalent to the fact that the KPZ relation sends $x = 0$ to $\Delta = 0$, in other words, if the simple random walk covers most of the lattice in the Euclidean case (say for example covers most of the ball of radius $r^{1-o(1)}$ before exiting the ball of radius r) then it should be the same in the random lattice case.

SAW vs SRW. One of the keys to our result is that after discovering a certain part Q_n of the UIPQ which corresponds to the hull of a simple random walk (but could be the hull of a percolation cluster ...) then the unknown quadrangulation with a boundary $Q_\infty \setminus Q_n$ is independent of Q_n conditionally on the length of the boundary.

The boundary of $\partial(Q_\infty \setminus Q_n)$ can be seen as a self-avoiding loop surrounding the discovered part. Indeed, the annealed model of self-avoiding walk (SAW) on a quadrangulation is totally equivalent to the model of quadrangulation with a simple boundary, just zip the boundary or unzip the SAW, see [19]. Heuristically speaking, we see that locally the boundary of the range of a simple random walk on the UIPQ is, in a certain sense, close to a self-avoiding walk. This fact is conjectured in planar Euclidean geometry, but still open.

References

- [1] J. Ambjørn, B. Durhuus, and T. Jonsson. *Quantum geometry*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1997. A statistical field theory approach.
- [2] O. Angel. Scaling of percolation on infinite planar maps, i. *arXiv:0501006*.
- [3] O. Angel. Growth and percolation on the uniform infinite planar triangulation. *Geom. Funct. Anal.*, 13(5):935–974, 2003.
- [4] O. Angel and N. Curien. Percolations on infinite random maps. *In preparation*.
- [5] O. Angel and O. Schramm. Uniform infinite planar triangulation. *Comm. Math. Phys.*, 241(2-3):191–213, 2003.

- [6] M. T. Barlow. Which values of the volume growth and escape time exponent are possible for a graph? *Rev. Mat. Iberoamericana*, 20(1):1–31, 2004.
- [7] M. T. Barlow and T. Kumagai. Random walk on the incipient infinite cluster on trees. *Illinois J. Math.*, 50(1-4):33–65 (electronic), 2006.
- [8] I. Benjamini. Random planar metrics. *Proceedings of the ICM 2010*, 2010.
- [9] I. Benjamini and N. Curien. Ergodic theory on stationary random graphs. *arXiv:1011.2526*.
- [10] I. Benjamini and P. Papasoglu. Growth and isoperimetric profile of planar graphs. *Proc. Amer. Math. Soc.*, 139(11):4105–4111, 2011.
- [11] I. Benjamini and O. Schramm. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.*, 6:no. 23, 13 pp. (electronic), 2001.
- [12] J. Bertoin. *Random Fragmentations and Coagulation Processes*. Number 102 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2006.
- [13] A. A. Borovkov and K. A. Borovkov. *Asymptotic analysis of random walks*, volume 118 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2008. Heavy-tailed distributions, Translated from the Russian by O. B. Borovkova.
- [14] J. Bouttier and E. Guitter. Distance statistics in quadrangulations with a boundary, or with a self-avoiding loop. *J. Phys. A*, 42(46):465208, 44, 2009.
- [15] A. K. Chandra, P. Raghavan, W. L. Ruzzo, R. Smolensky, and P. Tiwari. The electrical resistance of a graph captures its commute and cover times. *Comput. Complexity*, 6(4):312–340, 1996/97.
- [16] P. Chassaing and B. Durhuus. Local limit of labeled trees and expected volume growth in a random quadrangulation. *Ann. Probab.*, 34(3):879–917, 2006.
- [17] D. Croydon and T. Kumagai. Random walks on Galton-Watson trees with infinite variance offspring distribution conditioned to survive. *Electron. J. Probab.*, 13:no. 51, 1419–1441, 2008.
- [18] N. Curien, L. Ménard, and G. Miermont. A view from infinity of the uniform infinite planar quadrangulation. *arXiv:1201.1052*.
- [19] N. Curien and G. Miermont. Uniform infinite planar quadrangulations with a boundary. *available on arxiv*.
- [20] P. G. De Gennes. La percolation : un concept unificateur. *La Recherche*, 7:919–927, 1976.
- [21] B. Duplantier. Conformal random geometry. In *Mathematical statistical physics*, pages 101–217. Elsevier B. V., Amsterdam, 2006.
- [22] B. Duplantier and K.-H. Kwon. Conformal invariance and intersections of random walks. *Phys. Rev. Lett.*, 61(22), 1988.
- [23] B. Duplantier and S. Sheffield. Duality and the Knizhnik-Polyakov-Zamolodchikov relation in Liouville quantum gravity. *Phys. Rev. Lett.*, 102(15):150603, 4, 2009.

- [24] Z. Gao and L. B. Richmond. Root vertex valency distributions of rooted maps and rooted triangulations. *European J. Combin.*, 15(5):483–490, 1994.
- [25] J. T. Gill and S. Rohde. On the Riemann surface type of random planar maps. *arXiv:1101.1320*.
- [26] H. Kesten. Subdiffusive behavior of random walk on a random cluster. *Ann. Inst. H. Poincaré Probab. Statist.*, 22(4):425–487, 1986.
- [27] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov. Fractal structure of 2D-quantum gravity. *Modern Phys. Lett. A*, 3(8):819–826, 1988.
- [28] M. Krikun. Local structure of random quadrangulations. *arXiv:0512304*.
- [29] T. Kumagai. Random walks on disordered media and their scaling limits. *40th Probability Summer School, St. Flour, July 4–17, 2010*, 2010.
- [30] G. F. Lawler, O. Schramm, and W. Werner. The dimension of the planar Brownian frontier is $4/3$. *Math. Res. Lett.*, 8(4):401–411, 2001.
- [31] J.-F. Le Gall. Uniqueness and universality of the Brownian map. *arXiv:1105.4842*.
- [32] J.-F. Le Gall. Une approche élémentaire des théorèmes de décomposition de Williams. In *Séminaire de Probabilités, XX, 1984/85*, volume 1204 of *Lecture Notes in Math.*, pages 447–464. Springer, Berlin, 1986.
- [33] J.-F. Le Gall and L. Ménard. Scaling limits for the uniform infinite planar quadrangulation. *arXiv:1005.1738*.
- [34] R. Lyons, R. Pemantle, and Y. Peres. Conceptual proofs of $l \log l$ criteria for mean behavior of branching processes. *Ann. Probab.*, 23(3):1125–1138, 1995.
- [35] L. Ménard. The two uniform infinite quadrangulations of the plane have the same law. *Ann. Inst. H. Poincaré Probab. Statist.*, 46(1):190–208, 2010.
- [36] G. Miermont. The Brownian map is the scaling limit of uniform random plane quadrangulations. *arXiv:1104.1606*.
- [37] J. Neveu. Arbres et processus de Galton-Watson. *Ann. Inst. H. Poincaré Probab. Statist.*, 22(2):199–207, 1986.
- [38] G. Schaeffer. Conjugaison d’arbres et cartes combinatoires aléatoires. phd thesis. 1998.
- [39] S. Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *arXiv:1012.4797*.
- [40] Y. Watabiki. Construction of non-critical string field theory by transfer matrix formalism in dynamical triangulation. *Nuclear Phys. B*, 441(1-2):119–163, 1995.

Mathematics Department The Weizmann Institute Rehovot 76100, Israel	Département de Mathématiques et Applications Ecole Normale Supérieure, 45 rue d’Ulm 75230 Paris cedex 05, France
---	--

nicolas.curien@ens.fr
 itai@wisdom.weizmann.ac.il